Today.

Polynomials.

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Secret Sharing.

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Illustration: need at least 3 keys to open a bank vault

Other apps. we'll see: codes based on polynomials

TYPE OF CODE	REED-SOLOMON	LOW-DENSITY PARITY-CHECK (LDPC)	TURBO
APPLICA-TIONS	 DATA STORAGE (CD/DVD)	 WIFI, BROADCASTING	 CELLULAR (3G, 4G), SATELLITE COMMUNICATIONS

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Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

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Polynomials $P(x)$ with arithmetic modulo p: [1] $a_i \in \{0, \ldots, p-1\}$ and

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod{p},$$

for $x \in \{0, \ldots, p-1\}$.

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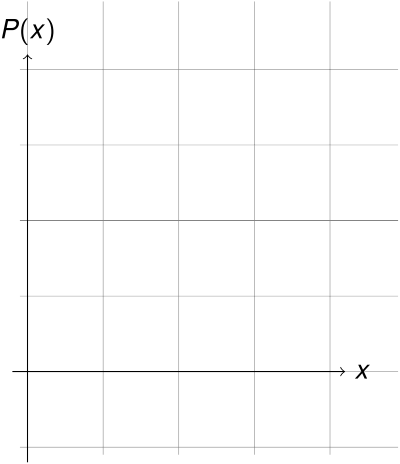

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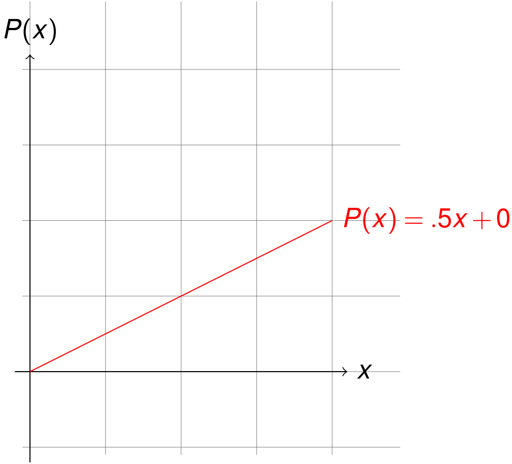

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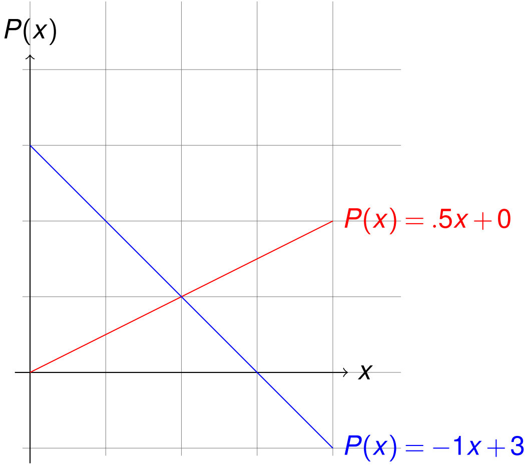

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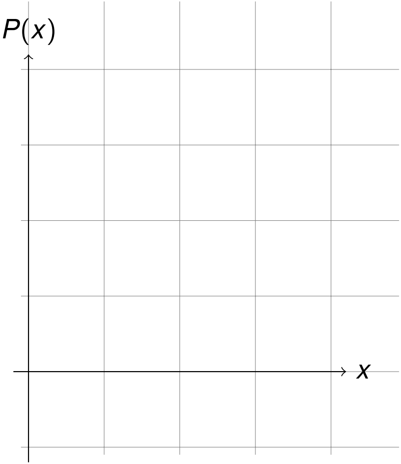

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0$

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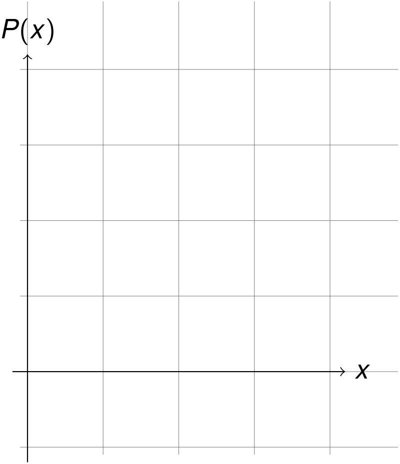

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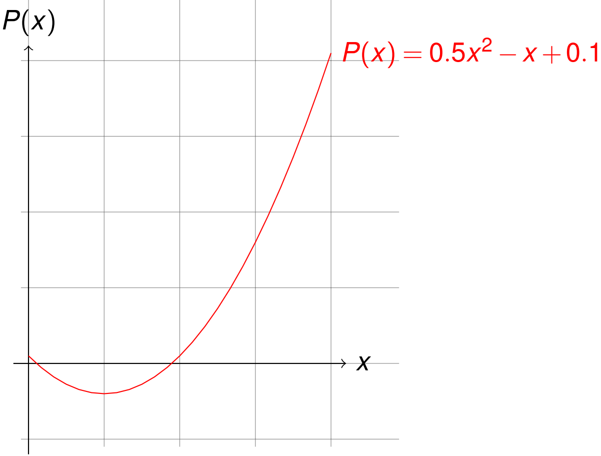

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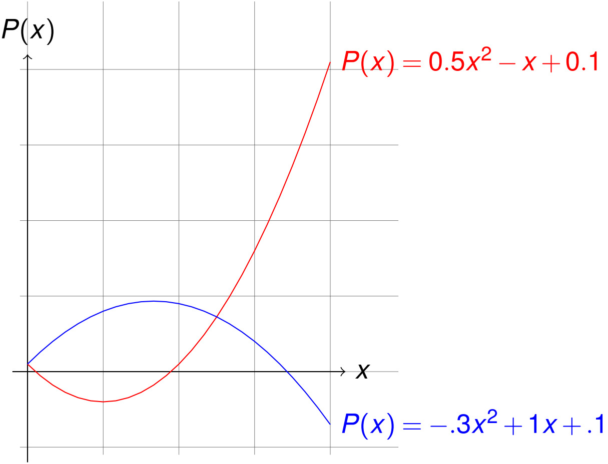

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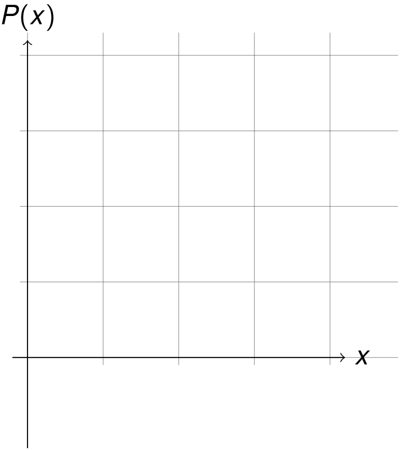

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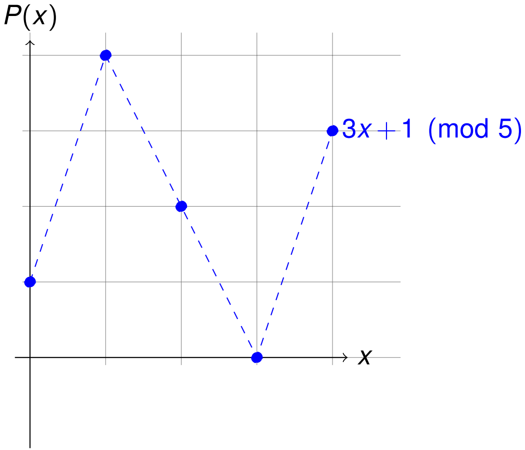

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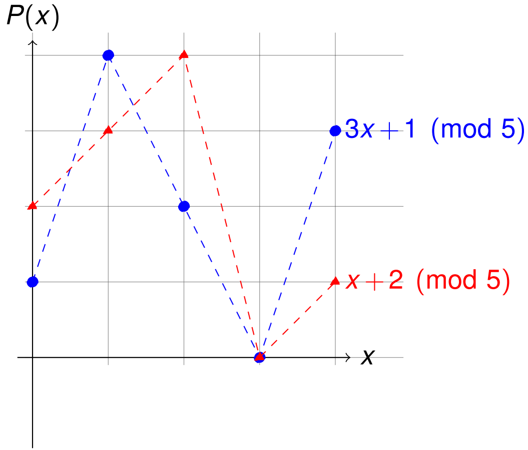

Finding an intersection.

$x + 2 \equiv 3x + 1 \pmod{5}$

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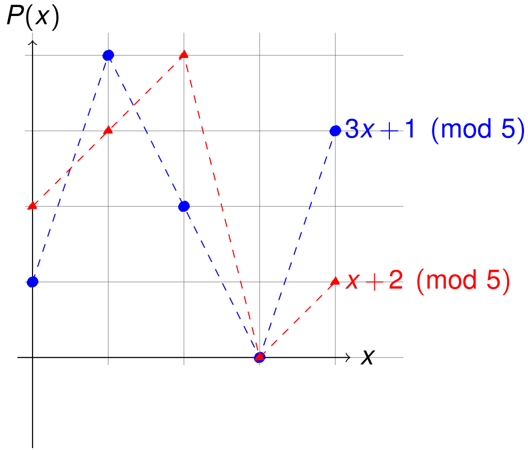

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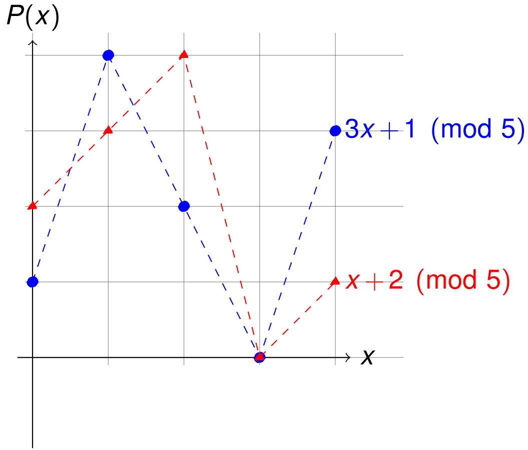

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Good when modulus is prime!!

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Fact: There is exactly 1 polynomial having degree $\leq d$ containing $d+1$ points. [2]

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Two points specify a line. Three points specify a parabola.

Modular Arithmetic Fact: There is exactly 1 polynomial having degree $\leq d$ (with arithmetic modulo prime p) containing $d+1$ pts.

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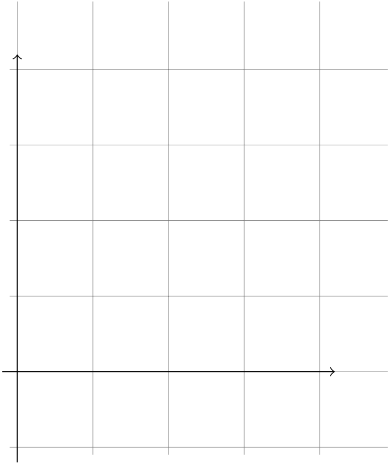
3 points determine a parabola.



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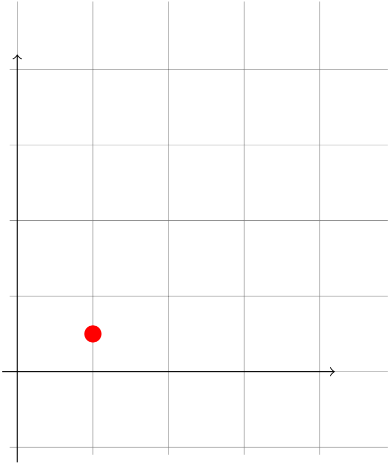

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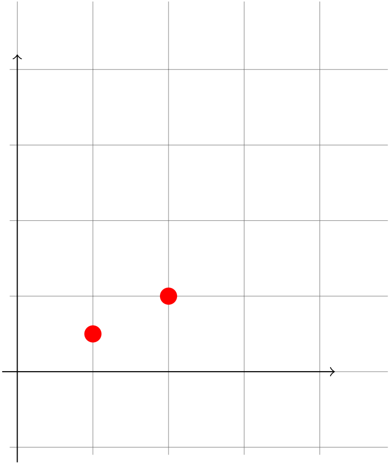

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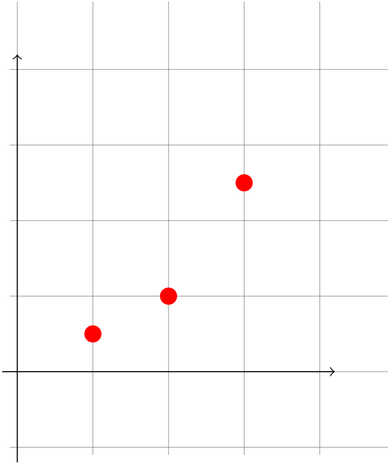

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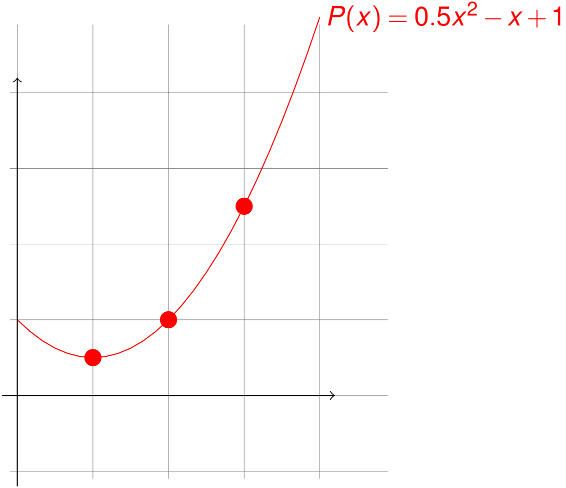

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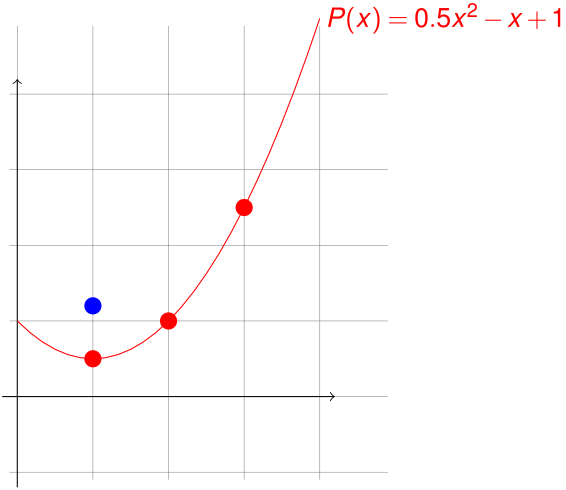

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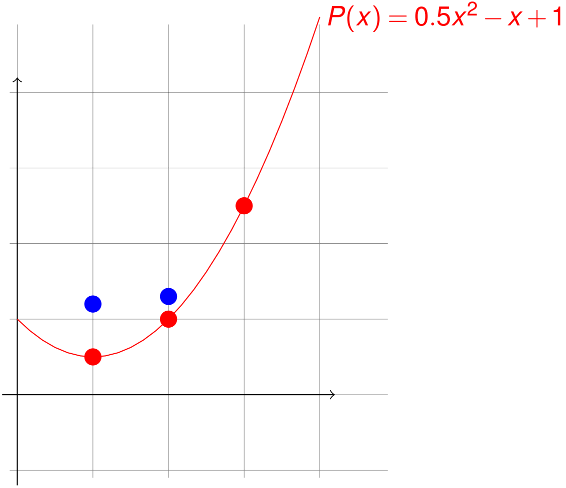

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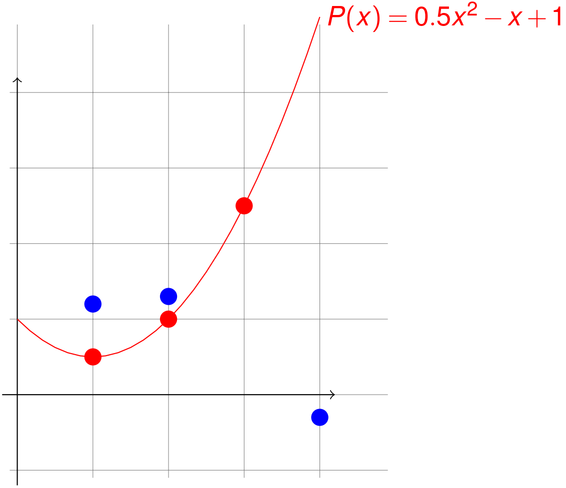

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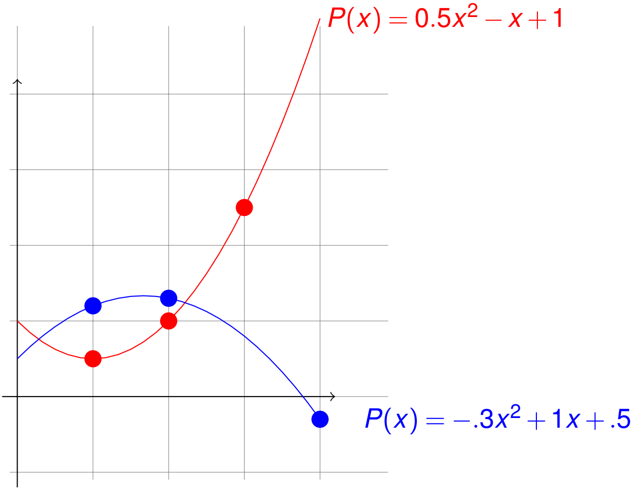

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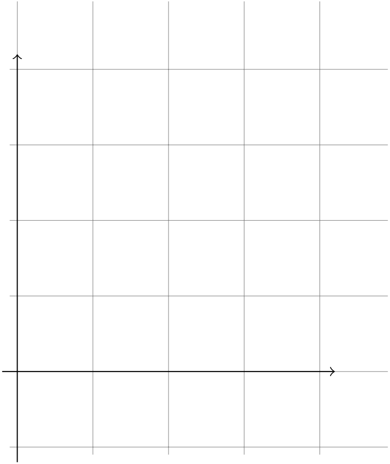

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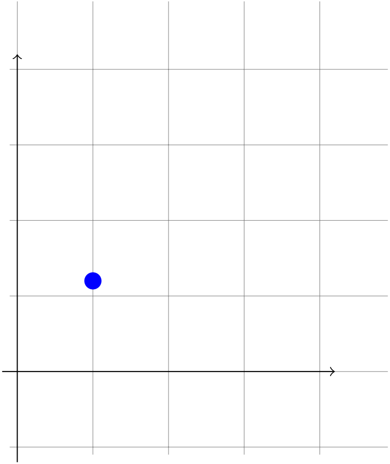

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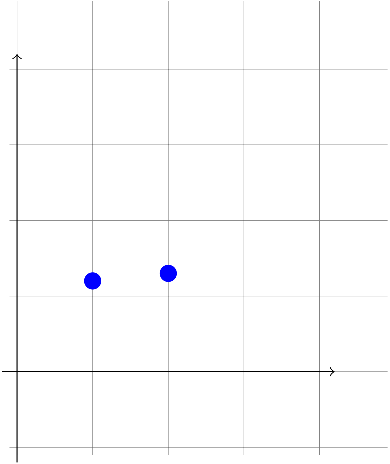

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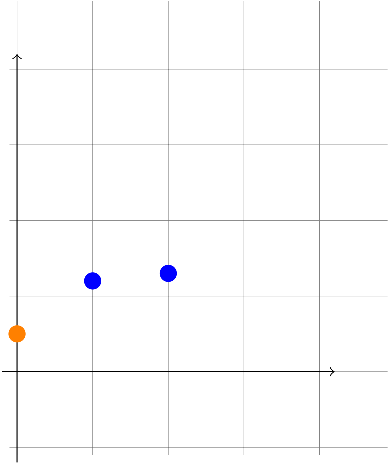

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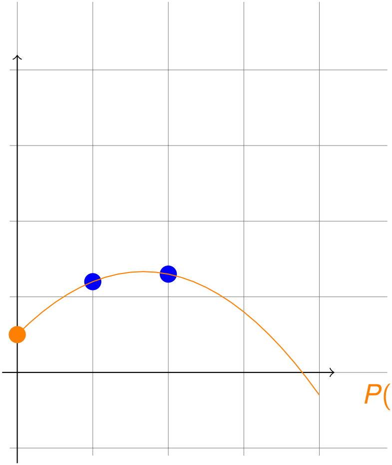

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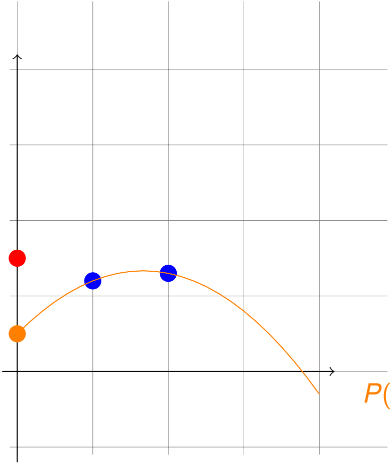

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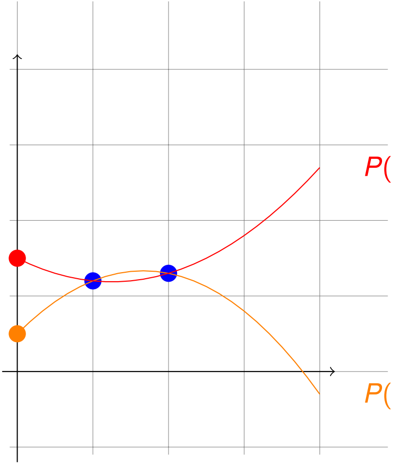

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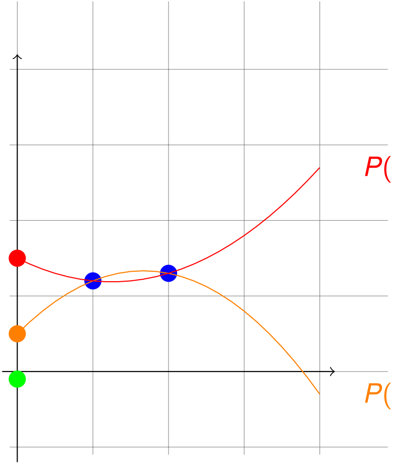

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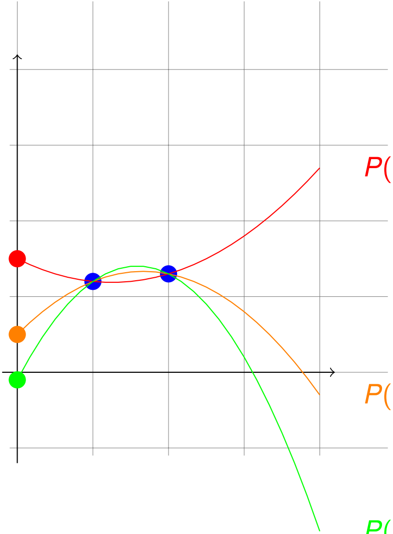

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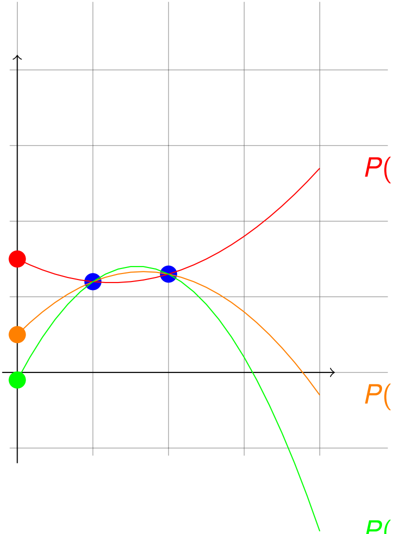

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Knowing k pts $\implies$ only one $P(x)$

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So polynomial is $2x^2 + 1x + 4 \pmod 5$

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We will work with polynomials with arithmetic modulo p.

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For set of x-values, $x_1, \ldots, x_{d+1}$.

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Construction proves the existence of a polynomial!

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Construction proves the existence of the polynomial!

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That is, $P(x) = (x - a)Q(x) + r$

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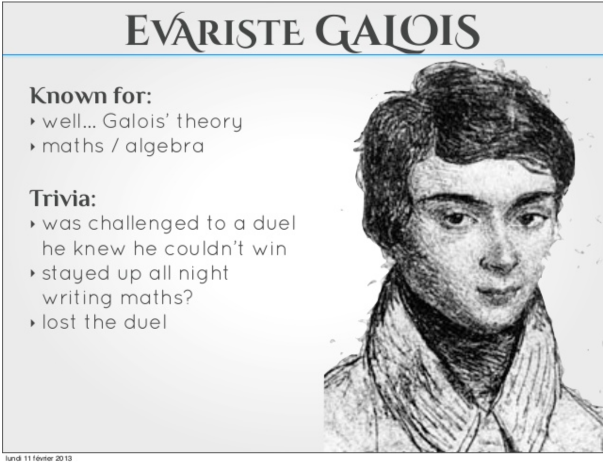
Arithmetic modulo a prime m is a **finite field** denoted by F_m or $GF(m)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

History lesson: Evariste Galois (1811-1832)



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Évariste Galois

From Wikipedia, the free encyclopedia

"Galois" redirects here. For other uses, see Gallois (disambiguation).

Évariste Galois (French: [evaʁist ɡaˈlwa]; 25 October 1811 – 31 May 1832) was a French mathematician born in Bourg-la-Reine. While still in his teens, he was able to determine a necessary and sufficient condition for a polynomial to be solvable by radicals, thereby solving a problem standing for 350 years. His work laid the foundations for Galois theory and group theory, two major branches of abstract algebra, and the subfield of Galois connections. He died at age 20 from wounds suffered in a duel.

Évariste Galois

A portrait of Évariste Galois aged about 15

Born	25 October 1811 Bourg-la-Reine, French Empire
Died	31 May 1832 (aged 20) Paris, Kingdom of France
Nationality	French
Alma mater	École préparatoire (no degree)
Known for	Work on the theory of equations and Abelian integrals
Scientific career	
Fields	Mathematics

Signature

Contents [hide]

Life [edit]

Early life [edit]

Galois was born on 25 October 1811 to Nicolas-Gabriel Galois and Adélaïde-Marie (born Demante).[1] His father was a Republican and was head of Bourg-la-Reine's liberal party. His father became mayor of the village after Louis XVIII returned to the throne in 1814. His mother, the daughter of a jurist, was a fluent reader of Latin and classical literature and was responsible for her son's education for his first twelve years. At the age of 10, Galois was offered a place at the college of Reims, but his mother preferred to keep him at home.

In October 1823, he entered the Lycée Louis-le-Grand, and despite some turmoil in the school at the beginning of the term (when about a hundred students were

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(Almost) the same as what is missing: one $P(i)$.

Runtime.

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Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k - 1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

A bit more counting.

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Infinite number for reals, rationals, complex numbers!